Sample Math 2002 Final

No calculators or other aids are allowed. Be sure to write your name and student ID# on the booklet provided. Questions 2 and 3 are worth 4 points, and all other questions are worth 6. You have three hours. Good luck!

1. (a) State Fubini's theorem for double integrals.
   (b) State Green's theorem.
   (c) State the divergence theorem.

2. Evaluate the following integrals:

   (a) The triple integral

   $$\int_0^1 \int_0^y \int_x^1 6xyz \, dz \, dx \, dy$$

   (b) The line integral $\int_C \mathbf{F} \cdot dr$, where

   $$\mathbf{F}(x, y, z) = (x + y)\mathbf{i} + (y - z)\mathbf{j} + (z^2)\mathbf{k}$$

   and C has parametrization $r(t) = t^2\mathbf{i} + t^3\mathbf{j} + t^2\mathbf{k}$.

3. Find the area of the shape bounded by the curves $x = y^2+2$, $x+3y = 0$.

4. Evaluate the triple integral $\int_E z \, dV$, where E is the part of the sphere $x^2 + y^2 + z^2 = 4$ which is in the first octant $(x \geq 0, y \geq 0, z \geq 0)$.

5. Use the change of variables $u = x + y$, $v = x - y$ to evaluate the double integral

   $$\int_D e^{\frac{x+y}{x-y}} \, dA$$

   where D is the region bounded by the curves $y = 0$, $x - y = 2$, $x = 0$, and $x - y = 1$.

6. For the vector field $\mathbf{F}(x, y, z) = (2xz + y^2)\mathbf{i} + 2xy\mathbf{j} + (x^2 + 3z^2)\mathbf{k}$,

   (a) Show that $\mathbf{F}$ is conservative.

   (b) Find a function f so that $\triangledown f = \mathbf{F}$.

   (c) Evaluate the line integral $\int_C \mathbf{F} \cdot dr$, where C has parametrization $x = t^2, y = t + 1, z = 2t - 1$, $0 \leq t \leq 1$.

7. Let E be the cylinder $x^2+y^2 = 4$, which is bounded by $z = 0$, $z = 3$. Let S be the surface of E, oriented inwards. Evaluate the surface integral $\int_S \mathbf{F} \cdot dr$, where $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$.

8. Solve the initial value problem $y'' + 6y' + 9y = e^{2x}$, $y(0) = 2, y'(0) = 0$.

9. As a fish swims through a stream looking for food to eat, it needs to counteract the work of the current. Find how much work the fish needs to counteract, if its path through the stream is given by the intersetion of the curves $y^2 + z^2 = 9$, $x + z = 9$, and the force of the stream is given by the vector field $\mathbf{F}(x, y, z) = y^2 \cos z\mathbf{i} + 2xy \cos z\mathbf{j} - 2xy^2 \sin z\mathbf{k}$.

10. **(Bonus question, +3 marks)** Say that $\mathbf{F}$ is *independent of path* if, whenever C_1 and C_2 are two paths with the same starting and ending points, $\int_{C_1} \mathbf{F} \cdot dr = \int_{C_2} \mathbf{F} \cdot dr$.

    Suppose $\mathbf{F}$ is a vector field of two or three dimensions. Prove that if $\int_C \mathbf{F} \cdot dr = 0$ for any closed path C, then $\mathbf{F}$ is independent of path.

Spherical co-ordinates:

$$x = r \sin\phi \cos\theta, \; y = r \sin\phi \sin\theta, \; z = r \cos\phi$$

the Jacobian for the spherical co-ordinates transformation is $r^2 \sin\phi$.